Journal of Applied Mechanics and Technical Physics, Vol. 42, No. 2, pp. 345–351, 2001

EFFECT OF INITIAL IMPERFECTIONS ON THE FLEXURAL EIGENVIBRATIONS OF CYLINDRICAL SHELLS

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UDC 539.3:534.1

The effect of small initial deviations from the ideal circular shape of a shell on the frequencies and modes of flexural eigenvibrations is studied with the use of the linear theory of thin shallow shells. It is assumed that the initial deviations are responsible for interaction between flexural and radial vibrations of the shell. The modal equations are derived by the Bubnov–Galerkin method. It is shown that the initial deviations from the ideal circular shape split the flexural vibration spectrum, and the fundamental frequency decreases compared to that of the ideal shell.

Introduction. In manufacturing shells, small initial deviations from the ideal shape, which are also called the initial imperfections, are inevitable. As a rule, these deviations are described by the function $w_0(x,y)$ (w_0 is the distance between the points on the middle surfaces of the real and ideal shells). It is well known that the function $w_0(x,y)$ strongly affects the stability of shells. The initial imperfections also affect the vibrations of shells, in particular, their eigenfrequencies which, like the critical loads, are the integral characteristics of rigidity.

Despite the fact that the effect of the parameter $w_0(x,y)$ on the flexural vibrations of shells has been studied in many papers, some fundamental problems still remain unsolved. It is believed that the initial deviations from the ideal circular shape increase the fundamental frequency compared to the case of an ideal shell [1–3]. However, this conclusion is in doubt. In the presence of the deviation $w_0(x,y)$, the resistance of the shell decreases, which implies that the fundamental frequency must decrease rather than increase.

The aim of this study is to clarify the above-mentioned contradiction and obtain a new solution that describes the effect of initial imperfections on the flexural eigenvibrations of thin-walled cylindrical shells.

Equations of Motion. Let a simply supported cylindrical shell of radius R, length l, and thickness h perform small flexural vibrations. The mathematical model is based on known equations of the linear theory of thin-walled shallow shells which have the following form [4] for an isotropic imperfect shell:

$$\frac{1}{E}\nabla^4\Phi = -L(w_0, w) - \frac{1}{R}\,\frac{\partial^2 w}{\partial x^2}, \qquad \frac{D}{h}\nabla^4 w = L(w_0, \Phi) + \frac{1}{R}\,\frac{\partial^2\Phi}{\partial x^2} - \rho\,\frac{\partial^2 w}{\partial t^2}. \tag{1}$$

Here ∇^4 is the Laplace operator, $w(x,y,t)$ is the radial dynamic deflection which is positive when directed toward the axis of the shell, $\Phi(x,y,t)$ is a stress function, $D = Eh^3/(12(1-\mu^2))$ is the flexural rigidity of the shell, E is Young's modulus, μ is Poisson's ratio, ρ is the density, t is the time, and $L = \partial^2/\partial^2 x(\partial^2/\partial y^2) + \partial^2/\partial y^2(\partial^2/\partial x^2) - 2\partial^2/\partial x\,\partial y(\partial^2/\partial x\,\partial y)$ is the differential operator.

In determining the integral characteristics of vibrations, in particular, the lower eigenfrequencies of thin isotropic shells, Eqs. (1) based on the Kirchhoff–Love hypothesis give results close to experimental data [2, 4].

Traditional Solution. For an ideal shell, each point on the circumference can be a vibration node. Experiments have shown that the initial imperfections eliminate this uncertainty and lead to interaction between the conjugate flexural modes [1, 2]. Generally, the location of the nodes of these modes does not

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0021-8944/01/4202-0345 $25.00

depend on the way of exciting the shell and is determined by the function $w_0(x, y)$. Therefore, it is generally believed that in the linear formulation, the dynamic deflection of an imperfect shell can be described in the first approximation by the expression [2]

$$w(x, y, t) = h[a_1(t)\sin(\beta y) + a_2(t)\cos(\beta y)]\sin(\alpha x) \quad (\alpha = \pi/l, \quad \beta = n/R). \tag{2}$$

The conjugate flexural modes $\sin(\beta y)\sin(\alpha x)$ and $\cos(\beta y)\sin(\alpha x)$ in (2) are the eigenmodes of the ideal shell and correspond to the same wave parameter n. It is known that these modes of the ideal shell correspond to the same eigenfrequency.

Let the shell have initial imperfections that correspond to the character of the wave formation upon its flexural vibrations:

$$w_0(x, y) = h[a_{10}\sin(\beta y) + a_{20}\cos(\beta y)]\sin(\alpha x). \tag{3}$$

We first solve the problem using the traditional finite-dimension model (2). Substituting (2) and (3) into the first equation in (1), we obtain an inhomogeneous differential equation whose solution yields a function that determines the dynamic stresses in the middle surface of the imperfect shell:

$$\begin{aligned}\Phi = E[&\Phi_0\sin(\alpha x)\sin(\beta y) + \Phi_1\sin(\alpha x)\cos(\beta y) + \Phi_2\sin(\beta y) + \Phi_3\cos(\beta y)\\ &+ \Phi_4\cos(2\alpha x)\sin(\beta y) + \Phi_5\cos(2\alpha x)\cos(\beta y) + \Phi_6\sin(\alpha x) + \Phi_7\cos(2\alpha x)\\ &+ \Phi_8\sin(2\beta y) + \Phi_9\cos(2\beta y) + \Phi_{01}x^2/2 + \Phi_{02}xy + \Phi_{03}y^2/2].\end{aligned} \tag{4}$$

The first ten coefficients in (4), which correspond to a particular solution of the differential equation, are given by

$$\begin{gathered}\Phi_0 = \frac{h\alpha^2}{R(\alpha^2+\beta^2)^2}\,a_1, \quad \Phi_1 = \frac{h\alpha^2}{R(\alpha^2+\beta^2)^2}\,a_2, \quad \Phi_2 = -\frac{h^2\alpha^2}{2\beta^2}\,a_{10}a_3, \quad \Phi_3 = -\frac{h^2\alpha^2}{2\beta^2}\,a_{20}a_3,\\ \Phi_4 = \frac{h^2\alpha^2\beta^2}{2(4\alpha^2+\beta^2)^2}\,a_{10}a_3, \quad \Phi_5 = \frac{h^2\alpha^2\beta^2}{2(4\alpha^2+\beta^2)^2}\,a_{20}a_3, \quad \Phi_6 = \frac{h}{\alpha^2 R}\,a_3,\\ \Phi_7 = \frac{h^2\beta^2}{16\alpha^2}\,(a_{10}a_1 + a_{20}a_2), \quad \Phi_8 = -\frac{h^2\alpha^2}{16\beta^2}\,(a_{10}a_2 + a_{20}a_1), \quad \Phi_9 = \frac{h^2\alpha^2}{16\beta^2}\,(a_{10}a_1 - a_{20}a_2),\end{gathered} \tag{5}$$

where $a_3(t) \equiv 0$ for the traditional solution. The last three terms in (4) enter the general integral of the equation $\nabla^4\Phi = 0$. As in [2, 3], these terms make it possible to satisfy "on average" the tangential boundary conditions

$$\frac{1}{2\pi R}\int_0^{2\pi R} \sigma_x\,dy = \frac{1}{2\pi R}\int_0^{2\pi R} \tau\,dy = 0 \qquad \text{for} \quad x = 0, \quad x = l, \tag{6}$$

where $\sigma_x = \partial^2\Phi/\partial y^2$ and $\tau = -\partial^2\Phi/\partial x\,\partial y$ are the normal and shear stresses, respectively.

Using the continuity condition for the circumferential displacement $v(x, y, t)$

$$\int_0^{2\pi R} \frac{\partial v}{\partial y}\,dy = \int_0^{2\pi R} \Big[\frac{1}{E}\Big(\frac{\partial^2\Phi}{\partial x^2} - \mu\,\frac{\partial^2\Phi}{\partial y^2}\Big) + \frac{w}{R} - \frac{\partial w_0}{\partial y}\,\frac{\partial w}{\partial y}\Big]dy = 0 \tag{7}$$

and boundary conditions (6), we find $\Phi_{01} = h^2\beta^2(a_{10}a_1 + a_{20}a_2)/4$ and $\Phi_{02} = \Phi_{03} = 0$.

Substituting (2)–(4) into the second equation in (1) and employing the Bubnov–Galerkin method, we obtain the equations

$$\ddot{a}_1 + c_{11}a_1 + c_{12}a_{10}a_{20}a_2 = 0, \qquad \ddot{a}_2 + c_{21}a_{10}a_{20}a_1 + c_{22}a_2 = 0, \tag{8}$$

where the dots denote differentiation with respect to the dimensionless time $\tau = \lambda t$ (λ is the eigenfrequency), and the coefficients are given by

$$c_{11} = 1 + \frac{\varepsilon}{8\omega^2}\,[(3+\theta^4)a_{10}^2 + \theta^4 a_{20}^2], \qquad c_{22} = 1 + \frac{\varepsilon}{8\omega^2}\,[(3+\theta^4)a_{20}^2 + \theta^4 a_{10}^2],$$
$$c_{12} = c_{21} = \frac{3\varepsilon}{8\omega^2}, \qquad \varepsilon = \Big(\frac{n^2 h}{R}\Big)^2, \qquad \theta = \frac{\pi R}{nl}. \tag{9}$$

The squared dimensionless eigenfrequency of flexural vibrations of the ideal circular cylindrical shell is determined from the formula

$$\omega^2 = \frac{\rho}{E}\,R^2\lambda^2 = \frac{\varepsilon(1+\theta^2)^2}{12(1-\mu^2)} + \frac{\theta^4}{(1+\theta^2)^2}.$$

Equations (8) are similar to those given in [2]. However, the coefficients (9) differ from those of [2], since the solution obtained in [2] does not satisfy the continuity condition for the displacement $v(x, y, t)$ (7).

The frequency equation following from (8) determines two eigenfrequencies. Their squared values are

$$\Omega_{01\,k}^2 = (\omega_{01\,k}/\omega)^2 = 1 + \varepsilon\theta^4(a_{10}^2 + a_{20}^2)/(8\omega^2),$$
$$\Omega_{02\,k}^2 = (\omega_{02\,k}/\omega)^2 = \Omega_{01\,k}^2 + 3\varepsilon(a_{10}^2 + a_{20}^2)/(8\omega^2), \tag{10}$$

where the subscript k denotes the traditional solution.

Thus, the initial imperfection (3) splits the flexural frequency spectrum of the shell. A frequency mismatch also occurs when one of the amplitudes of the initial imperfections vanishes. The first formula in (10) implies that the deviation $w_0(x, y)$ increases the fundamental frequency compared to the case of an ideal shell. This was also inferred in [2, 3]. However, this conclusion is wrong, which, in our opinion, is a result of the improper approximation of the dynamic deflection (2). Relation (2) does not describe the decrease in the generalized flexural rigidity of the shell because of the presence of initial deviations from a circular shape.

It is noteworthy that, for $\theta \to 0$, i.e., in the limiting case of an infinitely long imperfect shell (a ring under conditions of plane strain), formulas (10) are similar to those of [5]:

$$\Omega_{01\,k}^2 = 1, \qquad \Omega_{02\,k}^2 = 1 + 9(1-\mu^2)(a_{10}^2 + a_{20}^2)/2. \tag{11}$$

According to (11), the lower eigenfrequency $\Omega_{01\,k}$ of an infinitely long shell does not depend on the deviation $w_0(x, y)$, whereas the frequency $\Omega_{02\,k}$ exceeds severalfold the fundamental frequency of the ideal shell for a resulting amplitude of initial imperfections being of the order of the shell thickness.

New Solution. We refine the traditional finite-dimension model (2). It is assumed that owing to the initial deviations from a circular shape, not only the conjugate flexural modes interact, but also the nodes of these modes can shift in the radial direction. In other words, it is assumed that the deviation $w_0(x, y)$ is responsible for coupled flexural–radial vibrations (for an ideal shell, these vibrations are independent of one another). In (2), we introduce the additional coordinate $a_3(t)$ corresponding to the radial mode of vibration:

$$w(x, y, t) = h[a_1(t)\sin(\beta y) + a_2(t)\cos(\beta y) + a_3(t)]\sin(\alpha x). \tag{12}$$

To compare models (2) and (12), we solve the problem with the tangential boundary conditions satisfied "on average." Substituting (3) and (12) into (1) and solving equations of the shallow shell theory according to the scheme proposed by P. F. Papkovich, we arrive at the modal equations

$$\ddot a_1 + c_{11}a_1 + c_{12}a_{10}a_{20}a_2 + c_{13}a_{10}a_3 = 0,$$
$$\ddot a_2 + c_{21}a_{10}a_{20}a_1 + c_{22}a_2 + c_{23}a_{20}a_3 = 0, \qquad \ddot a_3 + c_{31}a_{10}a_1 + c_{32}a_{20}a_2 + c_{33}a_3 = 0, \tag{13}$$

where the coefficients c_{11}, c_{22}, and $c_{12} = c_{21}$ are determined from (9), and the other coefficients are given by

$$c_{33} = \frac{1}{\omega^2}\Big\{1 + \frac{\varepsilon\theta^4}{12(1-\mu^2)} + \frac{\varepsilon\theta^4}{8}\Big[2 + \frac{1}{(1+4\theta^2)^2}\Big](a_{10}^2 + a_{20}^2)\Big\},$$

$$c_{13} = c_{23} = -\frac{8\varepsilon^{1/2}}{3\pi\omega^2}\Big[1 + \frac{\theta^4}{(1+4\theta^2)^2}\Big], \quad c_{31} = c_{32} = -\frac{4\varepsilon^{1/2}}{3\pi\omega^2}\Big[1 + \frac{\theta^4}{(1+\theta^2)^2}\Big].$$

The insignificant deviation from the symmetry $c_{13} \neq 2c_{31}$ and $c_{23} \neq 2c_{32}$ is due to the fact that the tangential boundary conditions are satisfied "on average." The symmetry would occur if these conditions were satisfied exactly.

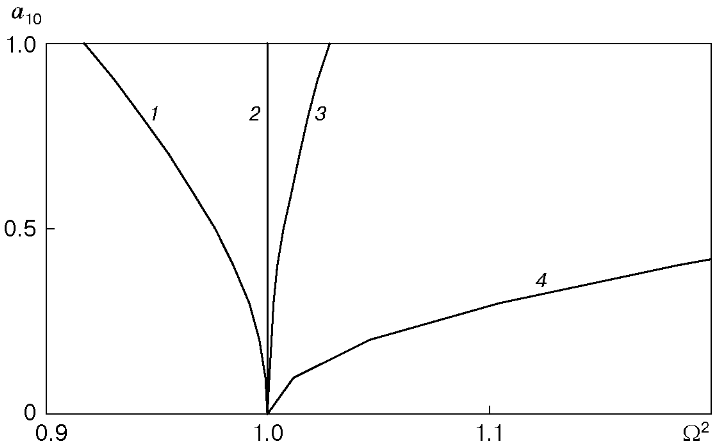

Fig. 1. Squared flexural eigenfrequency of the shell versus the initial-imperfection amplitude: curve 1 refers to the new solution Ω_{01}^2, curve 2 refers to the solution for the ideal shell (Ω^2), curve 3 refers to the new (Ω_{02}^2) and traditional ($\Omega_{01\,k}^2$) solutions, and curve 4 refers to the traditional solution $\Omega_{02\,k}^2$.

From the frequency equation following from (13), we find three eigenfrequencies: $\Omega_{01} < 1 < \Omega_{02} \ll \Omega_{03}$. The first two eigenfrequencies correspond to the essentially flexural vibrations of the imperfect shell, and the third eigenfrequency refers to the essentially radial vibrations. The squared second eigenfrequency is determined exactly:

$$\Omega_{02}^2 = 1 + \varepsilon\theta^4(a_{10}^2 + a_{20}^2)/(8\omega^2). \tag{14}$$

It should be noted here that $\Omega_{02} = \Omega_{01\,k}$.

Discussion of Results. Figure 1 shows the squared flexural eigenfrequency of the shell with the parameters $l/R = 0.6$ and $R/h = 200$ versus the amplitude of the initial imperfection a_{10} ($a_{20} = 0$). The calculations were performed for the number of circumferential waves $n = 10$ and Poisson's ratio $\mu = 0.3$. Curve 2 ($\Omega^2 = 1$) refers to the squared frequency of the ideal shell. Curves 1 and 3 refer to the new solutions Ω_{01}^2 and Ω_{02}^2, respectively. For comparison, Fig. 1 shows the squared frequencies $\Omega_{01\,k}^2 = \Omega_{02}^2$ (curve 3) and $\Omega_{02\,k}^2$ (curve 4) calculated by formulas (10) corresponding to the traditional model (2). It is clear from Fig. 1 that in the new solution, the difference between the eigenfrequencies of the essentially flexural vibrations of thin-walled shells is insignificant, which is supported by experimental data [1]. Owing to the fact that the frequencies Ω_{01} and Ω_{02} are close, an intense energy exchange between the conjugate modes occurs. In contrast, the traditional solution shows a significant eigenfrequency mismatch. It is also noteworthy that in contrast to the traditional solution, the new solution shows that the fundamental frequency of the imperfect shell is lower than that of the ideal shell.

For relatively long imperfect shells (for $\theta < 0.5$), from the frequency equation one can obtain the following approximate expression for the squared fundamental frequency:

$$\Omega_{01}^2 \approx \Omega_{02}^2[1 - 3\varepsilon(a_{10}^2 + a_{20}^2)/8]. \tag{15}$$

Formula (15) implies that as the amplitude of the initial imperfections increases, their effect on the fundamental frequency becomes more pronounced.

In the limiting case of an infinitely long imperfect shell, formulas (14) and (15) become $\Omega_{01}^2 \approx 1 - 3\varepsilon(a_{10}^2 + a_{20}^2)/8$ and $\Omega_{02}^2 = 1$, respectively, i.e., they are similar to those given in [5].

The frequency of the essentially radial vibrations of the shell Ω_{03} increases with the amplitude of imperfections.

The Effect of the Formulation of Boundary Conditions. It is known that in the case of an ideal shell of finite length, the satisfaction of the tangential boundary conditions "on average" leads to a considerable error in determining its dynamic characteristics. The tangential boundary conditions also have an effect on the vibrations of shells with initial imperfections. However, this effect has not been studied in detail because of the complexity of the problem. In this paper, we consider the following four variants of tangential boundary conditions with the use the refined finite-dimensional model (12):

$$\sigma_x = \tau = 0, \quad \sigma_x = v = 0, \quad u = \tau = 0, \quad u = v = 0 \quad \text{for} \quad x = 0, \quad x = l. \tag{16}$$

These conditions are assumed to be the same at the ends and they are satisfied exactly.

Substituting (3) and (12) into the first equation of (1), we obtain a differential equation from which the following stress function Φ is determined:

$$\begin{aligned}\Phi = E\{&\Phi_0 \sin(\alpha x)\sin(\beta y) + \Phi_1 \sin(\alpha x)\cos(\beta y) + \Phi_2 \sin(\beta y) + \Phi_3 \cos(\beta y) + \Phi_4 \cos(2\alpha x)\sin(\beta y)\\
&+ \Phi_5 \cos(2\alpha x)\cos(\beta y) + \Phi_6 \sin(\alpha x) + \Phi_7 \cos(2\alpha x) + \Phi_8 \sin(2\beta y) + \Phi_9 \cos(2\beta y) + \Phi_{01} x^2/2\\
&+ \Phi_{02} xy + \Phi_{03} y^2/2 + [\Phi_{10}\cosh(\beta x) + \Phi_{11}\sinh(\beta x) + \Phi_{12}\beta x\sinh(\beta x) + \Phi_{13}\beta x\cosh(\beta x)]\sin(\beta y)\\
&+ [\Phi_{20}\cosh(\beta x) + \Phi_{21}\sinh(\beta x) + \Phi_{22}\beta x\sinh(\beta x) + \Phi_{23}\beta x\cosh(\beta x)]\cos(\beta y)\\
&+ [\Phi_{30}\cosh(2\beta x) + \Phi_{31}\sinh(2\beta x) + \Phi_{32}2\beta x\sinh(2\beta x) + \Phi_{33}2\beta x\cosh(2\beta x)]\sin(2\beta y)\\
&+ [\Phi_{40}\cosh(2\beta x) + \Phi_{41}\sinh(2\beta x) + \Phi_{42}2\beta x\sinh(2\beta x) + \Phi_{43}2\beta x\cosh(2\beta x)]\cos(2\beta y)\}.\end{aligned} \tag{17}$$

The first ten coefficients in (17) are calculated from formulas (5), and the other coefficients are determined depending on the variant of tangential boundary conditions (16).

For $\sigma_x = \tau = 0$, we obtain

$$\Phi_{01} = 4\alpha^2\Phi_7, \qquad \Phi_{02} = \Phi_{03} = 0, \quad \Phi_{10} = -(\Phi_2 + \Phi_4),$$

$$\Phi_{11} = \frac{1}{\beta l + \sinh(\beta l)}\Big\{[1 - \cosh(\beta l)]\Phi_{10} - \frac{\alpha}{\beta}\,\beta l\Phi_0\Big\},$$

$$\Phi_{12} = \frac{1}{\beta l + \sinh(\beta l)}\Big\{ - \sinh(\beta l)\Phi_{10} + \frac{\alpha}{\beta}\,[1 + \cosh(\beta l)]\Phi_0\Big\},$$

$$\Phi_{13} = \frac{1}{\beta l + \sinh(\beta l)}\Big\{ - [1 - \cosh(\beta l)]\Phi_{10} - \frac{\alpha}{\beta}\sinh(\beta l)\Phi_0\Big\},$$

$$\Phi_{20} = -(\Phi_3 + \Phi_5), \quad \Phi_{21} = \frac{1}{\beta l + \sinh(\beta l)}\Big\{[1 - \cosh(\beta l)]\Phi_{20} - \frac{\alpha}{\beta}\,\beta l\Phi_1\Big\},$$

$$\Phi_{22} = \frac{1}{\beta l + \sinh(\beta l)}\Big\{ - \sinh(\beta l)\Phi_{20} + \frac{\alpha}{\beta}\,[1 + \cosh(\beta l)]\Phi_1\Big\},$$

$$\Phi_{23} = \frac{1}{\beta l + \sinh(\beta l)}\Big\{ - [1 - \cosh(\beta l)]\Phi_{20} - \frac{\alpha}{\beta}\sinh(\beta l)\Phi_1\Big\},$$

$$\Phi_{30} = -\Phi_8, \qquad \Phi_{31} = \frac{1 - \cosh(2\beta l)}{2\beta l + \sinh(2\beta l)}\,\Phi_{30},$$

$$\Phi_{32} = -\frac{\sinh(2\beta l)}{2\beta l + \sinh(2\beta l)}\,\Phi_{30}, \qquad \Phi_{33} = -\frac{1 - \cosh(2\beta l)}{2\beta l + \sinh(2\beta l)}\,\Phi_{30},$$

$$\Phi_{40} = -\Phi_9, \qquad \Phi_{41} = \frac{1 - \cosh(2\beta l)}{2\beta l + \sinh(2\beta l)}\,\Phi_{40},$$

$$\Phi_{42} = -\frac{\sinh(2\beta l)}{2\beta l + \sinh(2\beta l)}\,\Phi_{40}, \qquad \Phi_{43} = -\frac{1 - \cosh(2\beta l)}{2\beta l + \sinh(2\beta l)}\,\Phi_{40}.$$

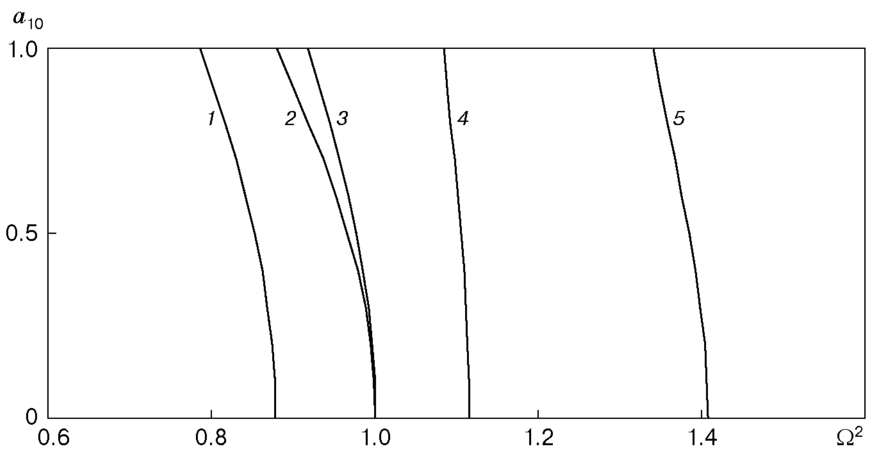

Fig. 2. Squared lower frequency of the shell versus the initial-imperfection amplitude: curve 1 refers to $\Omega^2_{01\,\sigma\tau}$ for the boundary conditions $\sigma_x = \tau = 0$, curve 2 to $\Omega^2_{01\,\sigma v}$ for $\sigma_x = v = 0$, curve 3 to Ω^2_{01} for the integral boundary conditions (6), curve 4 to $\Omega^2_{01\,u\tau}$ for the boundary conditions $u = \tau = 0$, and curve 5 to $\Omega^2_{01\,uv}$ for $u = v = 0$.

The coefficients for the other variants of the tangential boundary conditions at the ends of the shell have cumbersome expressions and here we omit them.

Substituting (3), (12), and (17) into the second equation of (1) and using the Bubnov–Galerkin method, we obtain equations similar to Eqs. (13). In these equations, the coefficients for $\sigma_x = \tau = 0$ are given by

$$c_{11} = 1 - \frac{8\theta^7}{\pi\omega^2(1+\theta^2)^4}\,\frac{\cosh(\pi/\theta)+1}{\sinh(\pi/\theta)+\pi/\theta}$$

$$+\frac{\varepsilon}{8\omega^2}\Big[(3+\theta^4)a_{10}^2 + \theta^4 a_{20}^2 - \frac{2\theta^5}{\pi}\,\frac{\cosh(2\pi/\theta)-1}{\sinh(2\pi/\theta)+2\pi/\theta}\,(a_{10}^2 + a_{20}^2)\Big],$$

$$c_{22} = 1 - \frac{8\theta^7}{\pi\omega^2(1+\theta^2)^4}\,\frac{\cosh(\pi/\theta)+1}{\sinh(\pi/\theta)+\pi/\theta}$$

$$+\frac{\varepsilon}{8\omega^2}\Big[(3+\theta^4)a_{20}^2 + \theta^4 a_{10}^2 - \frac{2\theta^5}{\pi}\,\frac{\cosh(2\pi/\theta)-1}{\sinh(2\pi/\theta)+2\pi/\theta}\,(a_{10}^2 + a_{20}^2)\Big],$$

$$c_{33} = \frac{p^2}{\omega^2} + \frac{\varepsilon\theta^4}{8\omega^2}\Big[2 + \frac{1}{(1+4\theta^2)^2} - \frac{512\theta^5(1+2\theta^2)^2}{\pi(1+4\theta^2)^4}\,\frac{\cosh(\pi/\theta)-1}{\sinh(\pi/\theta)+\pi/\theta}\Big](a_{10}^2 + a_{20}^2),$$

$$c_{13} = -\frac{8\varepsilon^{0.5}}{3\pi\omega^2}\Big\{1 + \frac{\theta^4}{(1+\theta^2)^2} + \frac{3\theta^8}{(1+\theta^2)^2(1+4\theta^2)}\Big[1 - \frac{4(1+2\theta^2)}{1+4\theta^2}\,\frac{\sinh(\pi/\theta)}{\sinh(\pi/\theta)+\pi/\theta}\Big]\Big\},$$

$$c_{12} = c_{21} = \frac{3\varepsilon}{8\omega^2}, \quad c_{23} = c_{13}, \quad c_{31} = \frac{1}{2}\,c_{13}, \quad c_{32} = \frac{1}{2}\,c_{23}, \quad p^2 = 1 + \frac{\varepsilon\theta^4}{12(1-\mu^2)}.$$

For the tangential boundary conditions denoted by the subscripts i and j, we find three dimensionless eigenfrequencies from the frequency equation: $\Omega_{01\,ij} < \Omega_{02\,ij} \ll \Omega_{03\,ij}$. The first two frequencies correspond to essentially flexural vibrations of the shell with the deviation $w_0(x,y)$, and the third frequency corresponds to essentially radial vibrations.

Figure 2 shows the squared lower dimensionless eigenfrequency of the shell with the parameters $l/R = 0.6$ and $R/h = 200$ versus the amplitude of initial imperfections a_{10} ($a_{20} = 0$). Calculations were performed for the number of circumferential waves $n = 10$ and Poisson's ratio $\mu = 0.3$. Curve 1 refers to the squared frequency $\Omega^2_{01\,\sigma\tau}$ for the boundary conditions $\sigma_x = \tau = 0$, curve 2 to $\Omega^2_{01\,\sigma v}$ for $\sigma_x = v = 0$, curve 3 to Ω^2_{01}

for the integral boundary conditions (6), the curve 4 to $\Omega^2_{01\,u\tau}$ for the boundary conditions $u = \tau = 0$, and the curve 5 to $\Omega^2_{01\,uv}$ for $u = v = 0$. One can see from Fig. 2 that, for the given parameters of the shell, the satisfaction of the tangential boundary condition "on average" leads to a considerable error in determining its dynamic characteristics. For all the variants of tangential boundary conditions, the initial imperfections decrease the fundamental frequency compared to the case of an ideal shell.

The calculation results show that, for a shell of relative length $\theta > 0.3$ whose ends can displace in the axial direction with and without imperfections, the satisfaction of the tangential boundary condition "on average" leads to significant overestimation of the exact values of the lower frequencies of essentially flexural vibrations. For the zero axial displacement of the ends, the satisfaction of the tangential boundary conditions "on average" leads to a considerable error even for $\theta > 0.2$. In this case, the exact values of the frequencies are underestimated.

Conclusions. It has been found that the initial deviations from the circular shape of a cylindrical shell significantly affect its flexural eigenfrequencies and lead to coupling of the conjugate flexural modes and interaction between the flexural and radial vibrations. The frequency spectrum is split, and the fundamental frequency decreases rather than increases compared to the case of an ideal shell.

The satisfaction of the tangential boundary conditions decreases the error in predicting the resonance conditions which can occur when a real shell is excited by external periodic or other dynamic loads.

In this paper, the effect of the initial imperfections on the eigenfrequencies has been estimated only preliminarily and qualitatively, since the solution of the problem is obtained in the first approximation.

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